The Longest Nesting Chain of Countries: A Certified Rigid-Motion Containment Result

(draft — author list omitted for review)

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Abstract

Which sequence of countries nests the deepest, each fitting strictly inside the previous as a rigid cardboard cutout? We pose this as a concrete computational-geometry puzzle: given a finite set of planar shapes, find the longest chain $s_1 \supset s_2 \supset \cdots \supset s_k$ in which each shape fits strictly inside its predecessor under *translation and rotation, with no reflection and no scaling* (shapes keep their true size). On the Natural Earth Admin-0 country polygons — each country projected in its own Lambert azimuthal equal-area frame and restricted to genuine countries and territories by an explicit deny-list — we compute a *certified* longest chain of **17 countries** at the finer 10 m source resolution (simplified to 0.5 km), spanning roughly 22.2 million km^2 down to the few-km^2 speck of Clipperton Island, with every link a sovereign state or a civilian-administered territory. (At the coarser, more heavily simplified 50 m tier the same pipeline certifies an 18-chain — the country filter does not bind there, whereas at 10 m it removes the denied entities the unfiltered 18-chain relied on, leaving 17; because the 10 m geometry is strictly closer to the true borders, we take it as the headline.) Our method composes a sound pairwise containment oracle into a directed acyclic graph (DAG) whose longest path is the answer. A *YES* verdict is backed by an explicit witness placement (rotation angle plus translation) and is exact; a *NO* verdict is sound *at the working grid resolution plus a per-rotation-interval Lipschitz buffer*, certifying that no placement exists with clearance above a stated tolerance τ, rather than being exact in the continuous limit. On a clean "cold" run (no cache reuse, float64 throughout) the optimistic and pessimistic longest-path bounds coincide — at 17 over the curated country set (18 over full Admin-0) — with *zero* uncertified pairs remaining, so the result is two-sided CERTIFIED. We also report the translation-only variant (**14 countries**, three links shorter, because rotation is genuinely needed: many countries seat inside their predecessor only well away from the projected orientation), and characterize how the answer depends on geometry resolution (a certified 17 at 10 m and 18 at 50 m). For the residual borderline gap we discuss two exact-arithmetic backstops as future work: the configuration-space method of Avnaim–Boissonnat, which would upgrade our resolution-sound NO verdicts to zero-tolerance exact, and Milenkovic's range-intersection branch-and-bound — the exact tightening of the very per-rotation-interval shape our oracle approximates with a Lipschitz buffer, hence the closest existing relative of our method, though itself finite-precision rather than exact. To our knowledge this is the first work to compose a sound rigid-motion containment oracle into a DAG longest-path search over a *set* of real non-convex polygons.

1 Introduction

Russian nesting dolls (*matryoshka*) nest as hollow three-dimensional shells; flatten the idea to the plane and it sharpens into a precise geometric question: among a fixed collection of flat shapes, how long a chain can we build in which each shape fits, as a rigid cutout, strictly inside the one before it? Replace the cutouts with the silhouettes of the world's countries and the puzzle couples a difficult

geometric primitive — deciding whether one non-convex polygon fits inside another under rotation and translation — with a combinatorial optimization over the resulting containment relation.

This paper answers the puzzle for the Natural Earth Admin-0 country dataset with an explicit certificate. The geometric primitive ("does polygon B fit inside polygon A under some rigid motion?") is the rotational polygon containment decision, which is known to be expensive: exact methods scale poorly and polygon containment is 3SUM/kSUM-hard [8, 9]. We therefore build a *sound* oracle — one whose verdicts never lie within a stated tolerance — rather than a heuristic one, stating exactly which verdict (YES vs. NO) carries which guarantee. We then lift the oracle to a global answer: "fits strictly inside" is a strict partial order, whose longest chain is the longest path in the corresponding directed acyclic graph.

Contributions.

- **A sound pairwise rigid-motion containment oracle** combining a cheap rotation-invariant pruning cascade, a rasterized feasibility test evaluated over all translations at once by FFT cross-correlation, and a Lipschitz branch-and-bound over rotation angle that produces (i) an explicit *witness* for YES and (ii) a resolution-plus-buffer certificate for NO at clearance tolerance τ (Section 3).
- **A composition into a certified longest chain.** We build the containment DAG, take its longest path by linear-time dynamic programming, and *certify* the answer by a criticality filter: we re-test at high precision only the unresolved pairs that could possibly lengthen the chain, and stop when the optimistic (treat-unresolved-as-YES) and pessimistic (treat-unresolved-as-NO) optima coincide (Section 4).
- **The certified country result.** A chain of **17 countries** under translation + rotation at the finer 10 m geometry, CERTIFIED (optimistic = pessimistic = 17), reported in full with per-link rotation angles and achieved clearances; the translation-only variant is **14**, and the coarser 50 m tier gives a certified 18 (Section 5). The chain is computed over an explicitly curated set of genuine countries and territories (Section 2.1).
- **A resolution-sensitivity characterization.** We solve the same instance at two geometry resolutions and with/without the country filter and show every cell is a two-sided certified optimum (Table 2); we explain *why* the optimum moves — the minimum-area inclusion floor is a provably monotone axis while boundary simplification is not (Section 5).
- **The limits**: the NO-soundness is resolution-plus-Lipschitz, not continuous-exact; the answer is projection-dependent and resolution-dependent; reflection is excluded by modeling choice (Section 7).

Why this is not trivial. A natural objection is that the whole exercise is elementary: "rotational containment is a solved problem — there is already a tolerance-based algorithm for it (Milenkovic's [1]), so just use that as the pairwise oracle and run DAG longest-path." The DAG half is indeed trivial, and deliberately so: our decomposition puts *all* the difficulty in the pairwise primitive, so that given a sound oracle the global answer is a linear-time dynamic program. The missing ingredient is a tolerance-based oracle that runs at scale, and supplying one is the contribution. Milenkovic's 1999 decision had, to our knowledge, no public implementation, is notoriously hard to build (robust finite-precision constructions, range-intersection, branch-and-bound), and where realized runs only at toy scale. It stays at toy scale because Milenkovic's range-intersection

B&B and the Avnaim–Boissonnat configuration-space method [3] ($O(m^3n^3 \log mn)$) are *polynomial in the vertex count*, and the cost bites twice — country silhouettes are large non-convex multi-polygons with holes, and the oracle must be called over the $O(n^2)$ ordered pairs (~ 250 countries $\Rightarrow \sim 90{,}000$ pairs). A method tractable for a handful of pieces cannot be invoked tens of thousands of times on polygons this large (and containment is 3SUM/kSUM-hard in general [8, 9]). Our oracle, by contrast, costs $O(A_\theta\, P^2 \log P)$ per pair — A_θ rotation branch-and-bound nodes, each one FFT cross-correlation over a $P \times P$ canvas that tests all translations at once — *independent of vertex count*: it is set by the resolution P (which fixes the tolerance τ) and the angular subdivision, not by how crenellated a coastline is. Trading an m^3n^3 dependence on boundary detail for a fixed grid cost is what makes the $O(n^2)$ sweep feasible. The contribution is thus a *sound* tolerance-based oracle fast enough for the whole pairwise sweep, plus the two-sided certification scheme that upgrades a tolerance-bounded per-pair oracle into a *certified* global optimum.

2 Problem definition

Shapes and rigid motions. A *shape* is a closed region of the plane bounded by a polygon, possibly with holes and possibly disconnected (a multipolygon). A *rigid motion* without reflection is a map $g_{\theta,t}(x) = R_\theta x + t$, where $R_\theta \in SO(2)$ is a rotation by angle θ and $t \in \mathbb{R}^2$ a translation. We allow rotation and translation, but *not* reflection and *not* scaling: shapes retain their true size and chirality.

Definition 1 (Strict containment) *For shapes A, B, we write $B \sqsubset A$ if there exists a rigid motion $g_{\theta,t}$ with $g_{\theta,t}(B) \subseteq \mathrm{int}(A)$, i.e. a rotated and translated copy of B lies in the (open) interior of A. A multipolygon B moves as one rigid body:* every *component must land inside A, and (since A's holes are not part of A) no part of B may overlap a hole of A.*

Chains and the containment DAG. The relation $\sqsubset$ is a strict partial order on a set of shapes $\mathcal{S} = \{s_1, \ldots, s_n\}$: it is irreflexive and transitive (a rigid motion of a rigid motion is a rigid motion). A *nesting chain* is a sequence $s_{i_1} \sqsupset s_{i_2} \sqsupset \cdots \sqsupset s_{i_k}$, i.e. a chain in this poset. Build the directed graph G with a vertex per shape and an arc $A \to B$ whenever $B \sqsubset A$. Because containment forces a strict area decrease ($\mathrm{area}(B) < \mathrm{area}(A)$ whenever $B \sqsubset A$ with $B \neq A$, as scaling is forbidden), G is acyclic, and a nesting chain is exactly a directed path in G.

Proposition 1 *The longest nesting chain over $\mathcal{S}$ is exactly the longest path in the containment DAG G: a nesting chain is a directed path and vice versa, so the two optima coincide.*

We consider two objectives on G: **links**, maximizing the number of countries (tie-broken by total area, since many chains share the maximum length), and **area**, maximizing the total summed area of the chain. Both are max-weight longest paths on the same DAG.

Instance. Our concrete instance is the Natural Earth Admin-0 Countries dataset (public domain), each country polygon reprojected into its *own* Lambert azimuthal equal-area (LAEA) frame centered on that country's representative point, then simplified with a topology-preserving tolerance. We solve it at two source tiers: the finer 10 m tier (simplified to 0.5 km), which we take as the headline because it is closer to the true borders, and the coarser 50 m tier (simplified to 2 km) as a cross-check. Per-country LAEA is an equal-area, low-distortion local frame; it is a modeling choice and the answer depends on it (Section 7).

2.1 What counts as a country

The Natural Earth Admin-0 layer is convenient and public-domain, but it is not a list of countries: alongside the world's sovereign states it ships a number of entities that no reasonable reader would call a country. Taking the layer wholesale lets a nesting chain pass through, for instance, the Baikonur Cosmodrome — a Russian-operated *lease* inside Kazakhstan — or an uninhabited reef. Our headline result is therefore computed over an explicitly curated subset, selected by a documented *deny-list* of 14 entities that we remove; everything else in Admin-0 is kept. The removed entities fall into two classes:

Leases and military installations (5): Baikonur Cosmodrome, US Naval Base Guantanamo Bay, Akrotiri Sovereign Base Area, Dhekelia Sovereign Base Area, and the British Indian Ocean Territory (Diego Garcia).

Uninhabited or undefined-boundary features (9): Bajo Nuevo Bank, Bir Tawil, Brazilian Island, Cyprus No Man's Area, Scarborough Reef, Serranilla Bank, Siachen Glacier, Southern Patagonian Ice Field, and the Spratly Islands.

We use an explicit editorial deny-list rather than filtering on Natural Earth's own `TYPE` field because that field misclassifies in both directions for our purpose. It buckets several *real, partially-recognized states* — Israel, Kosovo, Palestine, Western Sahara — as "Disputed" or "Indeterminate," so a naive `TYPE` filter would wrongly drop them; and it labels the Akrotiri and Dhekelia military bases as ordinary "Dependency," so a `TYPE` filter would wrongly keep them. The deny-list therefore *keeps* all sovereign states, all four partially-recognized states just named, and all civilian-administered dependencies (Greenland, Hong Kong, Niue, Montserrat, Ashmore & Cartier, Clipperton, the Falklands, Gibraltar, and the rest); Antarctica is also intentionally kept. These are dataset-classification decisions, made for the geometric puzzle, and carry no political endorsement of any entity's status. The filter is engaged by a `--countries-only` flag; the eight runs in Table 2 report the answer both with and without it.

3 The pairwise containment oracle

The oracle decides, for an ordered pair (A, B), whether $B \sqsubset A$. It has three layers: a cheap rotation-invariant pruning cascade, a rasterized feasibility test, and a Lipschitz branch-and-bound over the rotation angle.

3.1 Rotation-invariant pruning cascade

For each shape we precompute, once, a set of rigid-motion-invariant scalars: the area, the convex-hull area, the diameter (max pairwise hull distance), the minimum width (smallest slab over all hull-edge normals), the minimum enclosing radius menc, and certified one-sided bounds $[r_{\text{in}}^{\text{lo}}, r_{\text{in}}^{\text{hi}}]$ on the inscribed radius (largest inscribed disc), obtained from a Euclidean distance transform of the shape's own raster with a ± 2-pixel guard so the bounds remain sound. We then prune ordered pairs by the cascade of Algorithm 1.

Each cascade rule is *one-sided*: the diameter, width and hull-area tests are necessary conditions for containment and so can only certify NO; the inscribed/enclosing-disc test on line 1 certifies YES because a disc of radius $\text{menc}(B)$ contains B at any orientation and fits inside a disc of radius $r_{\text{in}}^{\text{lo}}(A)$ inside A. The raster-derived bounds use conservative (± 2-pixel) margins precisely so these shortcuts cannot flip a verdict.

Algorithm 1 Pruning cascade for ordered pair (A, B)

if $\mathrm{menc}(B) \le r_{\mathrm{in}}^{\mathrm{lo}}(A)$ **then return** FIT ▷ instant YES: B's enclosing disc fits in A's inscribed disc; holds at *every* rotation
if $\mathrm{diam}(B) > \mathrm{diam}(A)$ **then return** NOFIT ▷ too long to fit at any angle
if $\mathrm{width}(B) > \mathrm{width}(A)$ **then return** NOFIT ▷ too wide for A's narrowest slab
if $r_{\mathrm{in}}^{\mathrm{lo}}(B) > r_{\mathrm{in}}^{\mathrm{hi}}(A)$ **then return** NOFIT ▷ B's inscribed disc exceeds A's
if $\mathrm{hullarea}(B) > \mathrm{hullarea}(A)$ **then return** NOFIT
return UNDECIDED ▷ survivor: run the full test

3.2 Rasterized feasibility test over all translations

For a survivor pair we test feasibility at a fixed angle θ by rasterization. Choosing a pixel size s (below), we rasterize A and a rotated copy of B to binary masks. The key device is an FFT cross-correlation: with $\overline{A}$ the mask of A's complement (the "forbidden" cells) restricted to A's bounding box, the cross-correlation of $\overline{A}$ with the B-mask, evaluated at every integer translation that keeps B inside the canvas, gives in one shot the minimum number of B-pixels landing on forbidden cells over *all* grid translations. A value below $\frac{1}{2}$ means some translation places B entirely on allowed cells. Because any genuine fit must keep B inside A's bounding box, restricting to the canvas is sound (a B-mask larger than the canvas cannot fit and is reported as $+\infty$ overlap).

Resolution and tolerance. We *decouple* the working resolution from the clearance tolerance. The pixel size is $s = \max\big(\mathrm{maxdim}(A)/P,\ \tau_{\mathrm{req}}/6\big)$, where P is a target canvas resolution (default 700 px) and $\tau_{\mathrm{req}} = \max(\tau_{\mathrm{km}}, \tau_{\mathrm{rel}} \cdot \mathrm{diam}(B))$ is a requested clearance floor (defaults 1 km and 0.5% of $\mathrm{diam}(B)$). The clearance the pixel size certifies is $\tau = 6s$, and we use a rasterization guard of $g = 1.5\,s$. A hard canvas cap (default 2200 px) is a rare memory backstop; if it binds, the pair's effective τ is raised and logged.

Sandwich property. The rasterizer satisfies a verified sandwich: for any shape X,

$$\mathrm{mask}\big(X.\mathrm{buffer}(-1.5s)\big) \subseteq X \subseteq \mathrm{mask}\big(X.\mathrm{buffer}(+1.5s)\big).$$

Hence a fit certified between an *outer* (dilated) mask of B and an *inner* (eroded) mask of A is a true fit, and a no-fit certified between an inner mask of B and an outer mask of A is a true no-fit. The oracle always dilates the side that must contain and erodes the side that must be contained, so rasterization error can never manufacture a false verdict — it can only cost an extra subdivision.

3.3 Lipschitz branch-and-bound over rotation

Rotation is handled by branch-and-bound on angle intervals using a Lipschitz bound. Rotating B about a fixed center by $d\theta$ displaces any point by at most $R\,d\theta$, where R is the maximum distance from that center to any vertex of B. For an interval $[\theta_c - h,\ \theta_c + h]$ we therefore have, with $\mathrm{pad} = R\,h$ (in length units),

$$B(\theta_c).\mathrm{buffer}(-\mathrm{pad}) \subseteq B(\theta) \subseteq B(\theta_c).\mathrm{buffer}(+\mathrm{pad}) \qquad \forall\, \theta \in [\theta_c - h,\ \theta_c + h].$$

This yields two tests at the interval center θ_c:

- **YES (witness) at θ_c:** if the outer mask $\mathrm{mask}(B(\theta_c).\mathrm{buffer}(+g))$ fits inside the inner mask $\mathrm{mask}(A.\mathrm{buffer}(-g))$ at some grid translation, then B fits in A at angle θ_c — an explicit, exact witness.

- **NO over the whole interval:** if the inner mask $\text{mask}(B(\theta_c).\text{buffer}(-(\text{pad}+g)))$ fails to fit into $\text{mask}(A.\text{buffer}(+g+s))$ at *every* grid translation, then *no* angle in $[\theta_c - h, \theta_c + h]$ admits a fit (the eroded center mask is a subset of $B(\theta)$ for every θ in the interval, and the $+s$ slack on A covers sub-pixel translations).

Undecided intervals are subdivided; an interval is abandoned as a no-fit *at tolerance* once $\text{pad} \leq \tau/2$, i.e. once the residual ambiguity is smaller than the clearance we promise to resolve. We seed the search at evenly spaced angles plus the principal-axis alignment of A and B (and its 180° flip), test the cheaper NO bound first to prune intervals without running the YES FFT, and rotate a precomputed base raster of B (nearest-neighbour, displacement ≤ 0.71 px < 1.5 px guard) per node rather than re-rasterizing. In the translation-only variant ($k = 1$), the angle search collapses to a single decisive node at 0° with $\text{pad} = 0$.

3.4 Soundness

Lemma 1 (YES is exact) *If the oracle returns* Fit *with witness angle θ, then a rigid copy of B at angle θ and some translation lies strictly inside A. The verdict is exact: it is exhibited by a concrete placement and relies only on the sandwich property (the dilated B-mask, which contains $B(\theta)$, fits inside the eroded A-mask, which is contained in A).*

Lemma 2 (NO is sound at resolution + Lipschitz buffer) *If the oracle returns* NoFit *for (A, B), then there is no rigid motion placing B inside A with clearance $\geq \tau$ (i.e. such that $B.\text{buffer}(\tau)$ still fits), where $\tau = 6s$ is the clearance the working pixel size s certifies, augmented by the per-interval Lipschitz buffer $\text{pad} = Rh$ that bounds the grid-to-continuum gap over each rotation interval.*

The NO verdict is **sound at our resolution plus the Lipschitz bound**, *not* exact in the continuous limit. We do not claim that B provably fails to fit at every real angle and translation; we claim it fails to fit *with clearance at least τ*. A tangential, zero-clearance fit below τ is deliberately classified NO, which for physical cutouts (where a knife-edge fit is not a real fit) is the desired convention. Closing the gap between "no fit at clearance τ" and "no fit at clearance 0" is exactly the job of an exact-arithmetic oracle (Section 7).

3.5 Criticality classification of pairs

Each survivor pair, under a finite node budget, ends in one of three states: Fit (YES, witnessed), NoFit (NO at τ), or Unresolved (node budget exhausted before either certificate fired). The unresolved pairs are the borderline ones — congruent or near-tangent shapes — and they are isolated for the criticality filter of the next section.

Sub-resolution containees are Unresolved, not NO. When the containee B is so small that its raster collapses below the working resolution (its dilated mask degenerates relative to the pixel size s), the FFT test cannot be trusted to certify a NO: the rasterizer no longer faithfully represents B. We therefore classify any pair whose containee is sub-resolution as Unresolved rather than emitting a (potentially false) NoFit. This is the fix that keeps the two-sided certificate sound: a pessimistic chain built only from proven YES arcs can never be inflated by a NO verdict that was really a rasterization artifact, so optimistic = pessimistic genuinely certifies the optimum.

4 From oracle to chain

4.1 DAG construction with sound shortcuts

We add an arc $A \to B$ whenever the oracle (or a cascade shortcut) certifies $B \sqsubset A$. Two sound shortcuts cut the work. First, the instant-YES rule (Algorithm 1, line 1) adds many disc-containment arcs for free. Second, a *transitivity skip*: if B is already reachable from A through known YES arcs, then the arc $A \to B$ (whether present or not) cannot affect any longest path through this pair, so we skip its expensive test. We process candidate pairs in waves of increasing index gap so the transitivity skip fires maximally.

4.2 Longest path

The longest path is a single linear-time dynamic program over the DAG. Each vertex carries a weight *tuple* $(\text{primary}, \text{secondary})$ summed additively along a path and compared lexicographically: for the **links** objective the tuple is $(1, \text{area})$ (maximize count, tie-break by total area); for **area** it is $(\text{area}, 1)$ (maximize summed area, tie-break by length). Because every arc runs from larger to strictly smaller area, processing vertices in ascending area order solves every child before its parent. Both objectives are the same $O(V + E)$ recurrence.

4.3 Certifying the optimum

The unresolved pairs create an interval of possible optima. We bound it from both sides:

- **Optimistic** chain: treat every unresolved pair as a YES arc (upper bound on the longest path).
- **Pessimistic** chain: use only certified YES arcs (lower bound, every link proven).

If the optimistic and pessimistic optima *coincide* (in length for **links**, in value within a relative ϵ for **area**), the unresolved pairs provably cannot change the optimum and the answer is CERTIFIED. Otherwise we identify the unresolved pairs lying on the optimistic chain — the only ones that could matter — and re-test *just those* at the high precision budget, looping until the bounds meet or no further re-test is possible (Algorithm 2). This spends the expensive budget only where it can change the answer.

Algorithm 2 Criticality loop certifying the longest chain

resolve all survivors at the cheap node budget
loop
  $C_{\text{yes}} \leftarrow$ certified YES arcs; $U \leftarrow$ unresolved pairs
  $\chi_{\text{opt}} \leftarrow \textsc{LongestPath}(C_{\text{yes}} \cup U)$; $\chi_{\text{pess}} \leftarrow \textsc{LongestPath}(C_{\text{yes}})$
  if value(χ_{opt}) − value(χ_{pess}) $< \epsilon$ **then**
    return χ_{pess} ▷ CERTIFIED
  end if
  crit $\leftarrow$ unresolved pairs lying on χ_{opt} not yet tested at high precision
  if crit $= \emptyset$ **then**
    return χ_{pess} ▷ stalled: report uncertified blocking pairs
  end if
  re-test crit at the high node budget
end loop

5 Results

The certified 17-chain. On the 10 m instance under translation + rotation — our most accurate geometry — the longest nesting chain over genuine countries has **17 countries** and the result is **CERTIFIED**: the optimistic and pessimistic longest chains both have length 17, and the cold run leaves *zero* uncertified pairs. Here the country filter of Section 2.1 is *load-bearing*: the unfiltered 10 m Admin-0 layer admits an 18-chain, but that chain threads two non-countries — the Baikonur Cosmodrome (a Russian-operated lease) in its interior and Serranilla Bank (an uninhabited reef) at the very bottom. Removing the 14 denied entities (19,719 → 17,516 certified YES arcs) forces the longest path off them, and the certified optimum over genuine countries and territories is **17**. Every link is a sovereign state or a civilian-administered territory; the two smallest are Monaco and Clipperton Island (an uninhabited French Pacific atoll). The chain under the default **links** objective (tie-broken by total area) is

$$\begin{aligned}\text{Russia} \supset \text{Argentina} \supset \text{Niger} \supset \text{Kenya} \supset \text{Ivory Coast} \supset \text{Cambodia} \supset \text{Hungary}\\ \supset \text{Bhutan} \supset \text{El Salvador} \supset \text{Lebanon} \supset \text{Samoa} \supset \text{Dominica} \supset \text{Niue} \supset \text{Montserrat}\\ \supset \text{Norfolk Island} \supset \text{Monaco} \supset \text{Clipperton Island},\end{aligned}$$

with total chain area $\approx$ 22,199,456 km^2, detailed link-by-link in Table 1 and drawn as a stack of seated witness placements in Figure 1. It spans from Russia ($\approx$ 16.98 million km^2) down through Monaco ($\approx$ 17 km^2) to Clipperton Island ($\approx$ 4.5 km^2) — a more-than-three-million-fold area shrink.

Rotation is genuinely needed. Rotation is not cosmetic: forbidding it shortens the certified chain from 17 to 14 (the translation-only variant below), a loss of three links. Many links seat inside their predecessor only well away from the projected orientation — for instance Argentina inside Russia at $\approx 35.6°$, Kenya inside Niger at $\approx 107.3°$, Cambodia inside Ivory Coast at $\approx 263.3°$, and Bhutan inside Hungary at $\approx 141.0°$ — so rotation materially changes the achievable chain.

Translation-only variant. Forbidding rotation (translation only, the $k = 1$ collapse of the oracle) yields a certified chain of **14 countries** at the same 10 m geometry:

$$\begin{aligned}\text{Russia} \supset \text{Kazakhstan} \supset \text{Egypt} \supset \text{Iraq} \supset \text{Suriname} \supset \text{Sierra Leone} \supset \text{Lesotho}\\ \supset \text{Qatar} \supset \text{Luxembourg} \supset \text{Dominica} \supset \text{Niue} \supset \text{Montserrat} \supset \text{Nauru} \supset \text{Clipperton Island},\end{aligned}$$

Table 1: The certified rotation-allowed 17-chain at 10 m geometry, with per-link containment evidence: container area, achieved clearance τ (km), and the witness rotation angle. The "instant-YES" link (Monaco ⊃ Clipperton Island) is a disc-containment fit that holds at every rotation; the four deepest certified links sit at the $\tau = 0.5$ km tolerance floor. All figures are taken directly from the certified solver output (`run_rot_10m_countries/result.json`).

#	Container ⊃ containee	Area (km^2)	τ (km)	angle	kind
1	Russia ⊃ Argentina	16,979,778	34.62	35.62°	certified
2	Argentina ⊃ Niger	2,782,936	15.77	303.00°	certified
3	Niger ⊃ Kenya	1,181,351	7.21	107.25°	certified
4	Kenya ⊃ Ivory Coast	585,659	4.60	106.50°	certified
5	Ivory Coast ⊃ Cambodia	320,679	3.02	263.25°	certified
6	Cambodia ⊃ Hungary	181,037	2.46	15.00°	certified
7	Hungary ⊃ Bhutan	93,211	2.19	141.00°	certified
8	Bhutan ⊃ El Salvador	40,395	1.43	334.50°	certified
9	El Salvador ⊃ Lebanon	20,525	1.12	261.00°	certified
10	Lebanon ⊃ Samoa	9,998	0.78	87.00°	certified
11	Samoa ⊃ Dominica	2,775	0.62	105.00°	certified
12	Dominica ⊃ Niue	733	0.50	1.50°	certified
13	Niue ⊃ Montserrat	221	0.50	3.00°	certified
14	Montserrat ⊃ Norfolk Island	98	0.50	46.50°	certified
15	Norfolk Island ⊃ Monaco	41	0.50	69.00°	certified
16	Monaco ⊃ Clipperton Island	17	—	—	instant-YES
17	Clipperton Island	≈ 4.5			

with total area ≈ 21,393,912 km^2 (Figure 2). It bottoms out at Clipperton Island — the same innermost country as the rotation chain — but reaches it through three fewer links. That $14 \leq 17$ is not a coincidence: every translation-only fit is also a rigid-motion fit, so the no-rotation DAG is a subgraph of the rotation DAG and its longest path cannot be longer. The no-rotation chain leans heavily on disc-containment (roundish countries that fit at any orientation).

Run facts (cold runs). We report *cold* runs — no pairwise cache reuse, float64 arithmetic throughout — so every reported number is reproduced from scratch rather than replayed from a warm cache. The 10 m rotation run pruned the $\binom{n}{2}$ ordered pairs and built a containment DAG with **19,719** certified YES arcs (full Admin-0); it completed in ≈ 210 minutes (`runtime_min` = 210.24) and left **zero uncertified pairs**, so the optimistic and pessimistic longest paths coincide and the two-sided certificate pins the length with no borderline arcs outstanding. Applying the country filter reuses the same proven per-pair verdicts (the cached YES/NO certificates are geometry-keyed and re-used only within an identical geometry), so the filtered run is near-instant; the filter removes the 14 denied entities from the node set, dropping the rotation DAG to **17,516** arcs and the optimum from 18 to 17. The 10 m translation-only variant built **18,810** YES arcs in ≈ 3.3 minutes (`runtime_min` = 3.33), since the angle branch-and-bound collapses to one decisive node per pair; the filter drops it to **16,600** arcs and length 14. The coarser 50 m runs (the rotation cold run is 19,139 arcs in ≈ 57.7 minutes) give a certified 18/14 that the filter leaves unchanged. All eight runs report `uncertified_pairs` = [].

Country nesting chain — 17 deep (rotation allowed)
nesting order (outer → inner)
1. Russia 16,979,778 km²
2. Argentina 2,782,936 km²
3. Niger 1,181,351 km²
4. Kenya 585,659 km²
5. Ivory Coast 320,679 km²
6. Cambodia 181,037 km²
7. Hungary 93,210 km²
8. Bhutan 40,395 km²
9. El Salvador 20,525 km²
10. Lebanon 9,998 km²
11. Samoa 2,774 km²
12. Dominica 733 km²
13. Niue 221 km²
14. Montserrat 97 km²
15. Norfolk Island 41 km²
16. Monaco 17 km²
17. Clipperton Island 4 km²

Figure 1: The certified 17-country rotation-allowed nesting chain at 10 m geometry, drawn as concentrically seated rigid cutouts. Each inner country is placed at a concrete witness orientation and translation (found by the same FFT cross-correlation the certifier uses) that seats it strictly inside its container with zero spill. The stack runs from Russia (outermost) down to Clipperton Island (innermost).

Resolution sensitivity. The longest-chain length is not a single number but a function of the input geometry's resolution. We solve the same problem at two Natural Earth source tiers — 50 m (simplified to 2 km) and 10 m (simplified to 0.5 km) — crossed with rotation/no-rotation and with/without the country filter. Table 2 reports all eight runs; *every* cell is a two-sided certified optimum (optimistic = pessimistic, zero uncertified pairs). Two cells move under the country filter, in each case because the unfiltered 10 m optimum routed through a denied entity: the rotation chain drops from 18 to 17 and the no-rotation chain from 15 to 14 (the daggered cells; the caption names the entities involved). Because the 10 m geometry is strictly closer to the true borders, our headline numbers are its filtered results — rotation 17, translation-only 14 — with the coarser 50 m tier (18/14, unaffected by the filter) as the cross-check.

Why the resolution axes behave differently. The resolution dependence decomposes into two axes that behave differently. The *minimum-area inclusion floor* (the area below which a projected polygon is dropped) is a **monotone** axis: lowering it can only *add* nodes to the containment DAG (and the arcs incident to them), never delete an existing node or arc, so the longest path is non-decreasing as the floor drops. The *boundary-simplification tolerance* is **not** monotone: changing it re-shapes *every* retained polygon, altering its inscribed radius, width, and footprint, so a containment edge can both *appear* and *disappear* as simplification changes. The two resolutions in Table 2 differ on *both* axes at once (50 m/2 km vs. 10 m/0.5 km), so they are two distinct geometries rather than a clean refinement of one another: 50 m→ 18 and 10 m→ 17/18 do not contradict, because the finer tier both admits new micro-features *and* reshapes the existing ones. Notably, under the country filter the no-rotation answer is a *stable* 14 at both resolutions. This characterizes how the optimum depends on boundary detail rather than leaving it as an unquantified caveat; a fuller

Country nesting chain — 14 deep (translation only)
nesting order (outer → inner)
1. Russia 16,979,778 km²
2. Kazakhstan 2,714,328 km²
3. Egypt 1,000,861 km²
4. Iraq 437,300 km²
5. Suriname 145,120 km²
6. Sierra Leone 71,560 km²
7. Lesotho 30,119 km²
8. Qatar 11,152 km²
9. Luxembourg 2,608 km²
10. Dominica 733 km²
11. Niue 221 km²
12. Montserrat 97 km²
13. Nauru 29 km²
14. Clipperton Island 4 km²

Figure 2: The certified 14-country translation-only nesting chain at 10 m geometry, seated under the same witness-placement search but with every inner country held at its projected orientation (zero rotation). It reaches the same innermost country as the rotation chain (Clipperton Island) but through three fewer links. The stack runs from Russia (outermost) down to Clipperton Island (innermost). Taken directly from the certified solver output (`nesting_out_norot_10m_countries/result.json`).

sweep of the monotone floor axis is natural future work (Section 7).

Table 2: Certified longest-chain length as a function of geometry resolution and the country filter. Rows: rotation allowed vs. translation-only. Columns: the two source tiers, each shown unfiltered (full Admin-0) and countries-only. **Every cell is two-sided CERTIFIED** (optimistic = pessimistic, zero uncertified pairs). Bold marks the two headline results (the finer 10 m tier, countries-only). At the two daggered cells (†) the filter changes the answer because the unfiltered 10 m optimum routed through a denied entity — the Baikonur Cosmodrome lease and Serranilla Bank for rotation, Serranilla Bank for no-rotation. Lengths read directly from the eight `result.json` files.

	50 m (simplify 2 km)		10 m (simplify 0.5 km)	
	unfiltered	countries-only	unfiltered	countries-only
rotation	18	18	18	**17**†
translation-only	14	14	15	**14**†

6 Related work

Two-polygon rotational containment. Milenkovic [1] solves it by branch-and-bound over angle intervals, bounding each interval with the range-intersection construction $\bigcap_{\theta\in[\alpha,\beta]} B(\theta)$ and testing a translational inner-fit polygon (a linear program) at each leaf. It rests on robust finite-precision constructions — so it is robust but *not* exact in the zero-tolerance sense — and is practical only for a handful of pieces or limited angle ranges. It is nonetheless the closest existing relative of our oracle: its range intersection is the exact tightening of the very shape our oracle approximates with a Lipschitz buffer, which is why we single it out as a borderline-pair arbiter (Section 7). Milenkovic and Daniels [2] treat translational containment via mathematical programming. Avnaim and Boissonnat [3] give an exact configuration-space method for placement under translation and rotation in $O(m^3n^3 \log mn)$, but it has, to our knowledge, never been built for non-convex real-world polygons at scale. Chazelle [4] solved the convex two-polygon containment problem, and Chan and Hair [5] recently improved convex containment from quadratic to near-linear time — both convex only.

Packing and nesting. The nesting/packing literature surveyed by Bennell and Oliveira [7] concerns packing many pieces into a single container using no-fit-polygon constructions; it is heuristic and YES-oriented, providing no sound NO certificate. The "Nesting of Touching Polygons" line of work of Seemann, Stadler and Hellmuth [6] concerns *fixed-position* embedded polygons with no rigid motion.

Raster, Fourier, and Lipschitz methods. The three devices our oracle assembles are each individually well established, and we claim none of them as new. Testing fit by rasterizing shapes and intersecting pixel masks goes back to early raster nesting [10]; evaluating all relative translations at once by FFT cross-correlation is the standard primitive of correlation-based docking [11] and of spectral packing; and globally searching a rotation by Lipschitz branch-and-bound is the basis of guaranteed-optimal registration such as Go-ICP [12]. In each of these the discretization is a *heuristic* or an optimization energy, and the rotation search minimizes an *alignment error* — none yields a two-sided containment *decision* with an error guarantee. The nearest sound-from-raster idea is conservative rasterization [13], which gives a deliberately *one-sided* over-approximate occupancy guarantee for rendering and continuous collision detection; it has no notion of a witnessed YES

and no rotation. Our contribution is orthogonal: we promote the raster-FFT fit test to a *two-sided* certificate — an exact witnessed transform on YES, and a guaranteed NO down to a clearance τ that folds in both the pixel resolution and the Lipschitz rotation buffer — which neither the heuristic raster/FFT methods (no certificate) nor conservative rasterization (one-sided, translation only) provide.

Combinatorial backbone and hardness. Reducing a longest chain in a containment poset to a longest path in a DAG is elementary. On the lower-bound side, polygon containment is fine-grained-hard: Künnemann and Nusser [8] prove 3SUM/kSUM-style lower bounds for polygon containment and related problems (building on the $O(n^2)$-problems framework of Gajentaan and Overmars [9]), explaining why a practical sound pipeline, rather than a polynomial exact algorithm, is the right tool at scale.

Our position. Prior work decides the two-polygon problem or packs into one container. None, to our knowledge, composes a *sound* pairwise rigid-motion containment oracle into a DAG longest-path search to find the longest nesting chain over a *set* of real non-convex polygons. That composition, the explicit sound-NO framing, and the certified country result are our contributions.

7 Limitations and future work

NO-soundness is resolution-plus-Lipschitz, not continuous-exact. As stressed in Section 3.4, a NO verdict certifies "no fit with clearance $\geq \tau$," not "no fit at clearance 0." Zero-clearance tangent fits below τ are deliberately reported NO. This is the one and only gap between our pipeline and a zero-tolerance algebraic decision.

Exact-arithmetic backstop (future work). On this instance the cold run leaves no unresolved pairs, but in general the borderline pairs — congruent or near-tangent shapes whose fit/no-fit turns on clearances finer than τ — are precisely where a sharper rotational-containment oracle would close the residual gap. Two are relevant, and they differ in kind. Avnaim–Boissonnat [3] with exact arithmetic is genuinely zero-tolerance exact, but tractable only at toy scale — its configuration-space construction hits a cubic-in-the-vertex-count wall. Milenkovic's [1] range-intersection branch-and-bound is the architecturally closest to our pipeline — its per-interval intersection is the exact version of the Lipschitz buffer we use — and so the more immediately graftable, though, resting on robust finite-precision constructions, it *tightens* τ rather than removing it. The fine-grained hardness of polygon containment [8, 9] bounds what is achievable — which is exactly why the practical sound pipeline exists. Integrating either as a final arbiter for any borderline pairs would drive NO verdicts from "sound at τ" toward exact, without changing the present *certified* answer.

Projection-dependence. The answer depends on the choice of map projection. We use per-country LAEA, an equal-area local frame; a different projection (or a single global projection) would distort country shapes differently and could change the chain. The result is a statement about country shapes *as projected*, not about abstract platonic "true" shapes (there is no distortion-free planar representation of a curved-Earth region).

Dataset and simplification dependence. "Country shape" depends on the Natural Earth resolution (10 m for the headline) and on the topology-preserving simplification (0.5 km at 10 m). Table 2 and the accompanying analysis solve the instance at a second resolution and characterize

the dependence, separating the *monotone* minimum-area floor axis from the *non-monotone* simplification axis. A dense certified sweep of the monotone floor axis — mapping out exactly which micro-features enter the optimum at which area threshold, and whether $L(\varepsilon)$ plateaus as $\varepsilon \to 0$ — is natural future work, and is feasible precisely because every point is a two-sided certified optimum.

Choice of country set. The headline restricts Admin-0 to a curated set of genuine countries and territories via the deny-list of Section 2.1. That list is an explicit editorial choice; a reader who disagrees with any inclusion or exclusion can re-run with a different set, and the unfiltered columns of Table 2 bound how much the answer can move. We deliberately keep partially-recognized states and civilian-administered dependencies and exclude only leases, military installations, and uninhabited/undefined-boundary features; the choices are geometric-puzzle conventions, not political claims.

Reproducibility. The instance is the Natural Earth *Admin 0 – Countries* layer (public domain), fetched from the official NACIS CDN. The headline uses the 10 m cultural tier[1] simplified with a topology-preserving tolerance of 0.5 km; the coarser 50 m tier[2] simplified to 2 km serves as the cross-check. The exact source archives we validated against are pinned by SHA-256: the 10 m zip is `ce1ac7036499a0edd641fbc093cd209a98f96a49d2eca8480aaacad35138a7f6` and the 50 m zip is `5fed433373581fa648920435f937d95f2d3c0200e067409c6478dcdf1b853139`. We project each country into its own Lambert azimuthal equal-area frame centered on its representative point. The pipeline runs on CPython 3.14.5 with `shapely` 2.1.2, `geopandas` 1.1.3, `scipy` 1.17.1, `pyproj` 3.7.2 and `numpy` 2.4.6, using float64 arithmetic throughout. Countries are keyed for caching by their ISO A3 code, with a fallback fix: Natural Earth uses the truthy placeholder string `"-99"` as ISO A3 for entities without an assigned code, and eight distinct entities share it, so keying on the raw code would silently conflate their cached verdicts. We therefore treat `"-99"` (and any missing code) as no-ISO and fall back to the country's unique display name, then enforce global key uniqueness. The cold runs in this paper use a fresh (empty) pairwise cache.

Reflection excluded by choice. We forbid reflection (a cutout is not flipped over), matching the physical-cutout intuition. Allowing reflection would only enlarge the arc set and could lengthen the chain; it is a modeling choice, not a limitation of the method, and the oracle extends to it by adding the mirrored copy of B.

Multipolygons and holes. A multipolygon country moves as a single rigid body and must land entirely inside the container, and nested shapes may not overlap the container's holes. These semantics are enforced by the rasterization (holes are not filled) and affect a handful of island and enclave countries.

8 Conclusion

We posed the longest nesting chain of countries as a clean computational-geometry problem — the longest path in a rigid-motion containment DAG — and solved it with a pipeline whose verdicts carry explicit guarantees: YES is an exact witnessed placement, NO is sound at the working resolution plus a Lipschitz buffer down to a stated clearance τ. Composing this sound oracle with

[1] `https://naciscdn.org/naturalearth/10m/cultural/ne_10m_admin_0_countries.zip`

[2] `https://naciscdn.org/naturalearth/50m/cultural/ne_50m_admin_0_countries.zip` (Natural Earth v5.1.x release).

a criticality filter, we certify a longest chain of **17 countries** under translation + rotation at our most accurate (10 m) geometry (**14** for translation only), over a curated set of genuine countries and territories, with the optimistic and pessimistic optima coinciding at 17 and the cold run leaving zero borderline pairs outstanding. We further show the answer is a characterized function of geometry resolution rather than a fragile single number — the coarser 50 m tier gives a certified 18 — with each cell of the resolution table itself two-sided certified. The sound-NO framing, the oracle-to-DAG composition, the certified country answer, and the certified resolution-sensitivity characterization are, to our knowledge, new. Closing the residual borderline gap with an exact-arithmetic backstop, and a dense certified sweep of the inclusion-floor axis, are natural next steps.

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